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Decay to Equilibrium in Random Spin Systems on a Lattice. II

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We study the continuous spin systems on a $d \geqslant 3$-dimensional lattice with random ferromagnetic interactions of finite range. We show that, if the temperature is sufficiently high and the probability of interaction to be large is small enough, the almost sure decay to equilibrium has a subexponential upper bound.

KEY WORDS: Interacting random processes; dynamics of disordered systems; interacting particle systems; disordered systems.

1. INTRODUCTION

In ref. 3 we studied the stochastic dynamics of random spin systems on a lattice. In particular, we showed that for discrete as well as continuous-spin systems, if the random couplings are allowed to take on sufficiently large values, the generator of stochastic dynamics has no spectral gap with probability one even at high temperatures. As a consequence one concludes that the decay to equilibrium for local observable cannot be exponentially fast. Moreover, in that paper, under mild natural assumptions, we showed that, on a two-dimensional lattice, one has an almost sure stretched exponential upper bound on the decay to equilibrium in the uniform norm for continuous- as well as discrete-spin systems with random finite-range interactions. By this we improved the previous results of ref. 7, where a similar estimate was obtained for one-dimensional systems, and the results of an excellent

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paper[2] where a general but weaker upper bound was proven for discrete-spin systems with finite-range random interactions in any dimension. In this short review, we consider the continuous-spin random ferromagnet with finite range of interaction in a $d \geqslant 3$ lattice. We will state here the main results we are able to derive via methods similar to that we used in ref. 3. These methods are still efficient since they allow us to prove that, at high temperatures, one has faster decay to equilibrium than the one obtained in ref. 2. Our strategy is based on application of logarithmic Sobolev inequalities. Indeed, we obtain, for almost all interaction, bounds on the growth of the logarithmic Sobolev coefficient in finite (but large) sets with the size of these sets. In dimension $d \geqslant 3$, our estimates are growing too fast to ensure a stretched exponential decay as in ref. 3 and even any decay in the general setting (in view of the state of the art). Nevertheless, for ferromagnetic interactions, we can use a clever argument of ref. 6 (which extends the one of Holley's) to deduce from these basic estimates an upper bound on the decay to equilibrium. The reader is invited to read ref. 4 for the proofs of these results.

2. SETTING

Let $\mathbb{Z}^d$ be a $d \geqslant 3$-dimensional integer lattice with a metric $d(i, j) \equiv \sum_{\alpha=1}^{d} |i^\alpha - j^\alpha|$, $j, i \in \mathbb{Z}^d$. By $\mathscr{F}$ we denote the family of all finite subsets of the lattice. Let $\mathscr{F}_0 \equiv \{\Lambda_n \in \mathscr{F}\}_{n \in \mathbb{N}}$ be a countable exhaustion of the lattice, i.e., an increasing sequence of finite sets invading all the lattice.

We restrict ourselves to continuous spins, i.e., we will consider spins taking values in a smooth, compact, connected (finite-dimensional) Riemannian manifold M. For simplification, we will assume in the following that $\mathbf{M}$ is a compact subset of $\mathbb{R}^n$, for some integer number n. In this context, we will call the configuration space of the infinite spin system on the lattice $\mathbb{Z}^d$ the product space $\Omega \equiv \mathbf{M}^{\mathbb{Z}^d}$ with a Borel σ-algebra Σ given by the product topology.

In this setting, we would like to consider systems of particles in interaction via a random potential Φ. Let us denote $(\mathbb{J}, \mathscr{B}_j, \mathbb{E})$ the probability space on which this random potential lives. Then Φ is a family $(\Phi_X)_{X \in \mathscr{F}}$ consisting of measurable functions Φ_X: $\mathbb{J} \times \Omega \to \mathbb{R}$ such that:

- For every $X \in \mathscr{F}$, the function $\Phi_X(\mathbf{J}, \cdot)$ is continuous and Σ_X measurable (i.e., depends only on the coordinates $(\omega_j, j \in X)$),
- For every $i \in \mathbb{Z}^d$ we have

$$\sum_{\substack{X \in \mathscr{F} \\ X \ni i}} \|\Phi_X(\mathbf{J}, \cdot)\|_u < \infty \qquad \mathbb{E}\text{-a.s}$$

• The family of random variables $\{\Phi_X(\cdot, \omega), X \in \mathscr{F}\}$ on the probability space $(\mathbb{J}, \mathscr{B}_J, \mathbb{E})$ are mutually independent. Moreover, the random variables $\Phi_{X+j}(\cdot, T_j\omega)$ and $\Phi_X(\cdot, \omega)$ are identically distributed.

In this paper we restrict ourselves to potentials of finite range, i.e., we assume that there is a positive integer R such that $\Phi_X \equiv 0$ whenever $diam(X) \geqslant R$. Additionally we assume that the potential is twice differentiable, in the sense that for every $X \in \mathscr{F}$ the corresponding function $\Phi_X(\mathbf{J}, \cdot)$ is twice differentiable, for every $\mathbf{J} \in \mathbb{J}$.

Given a potential Φ, we define an interaction energy U_Λ in a finite set Λ by

$$U_\Lambda(\mathbf{J}, \omega) \equiv \sum_{\substack{X \in \mathscr{F} \\ X \cap \Lambda \neq \varnothing}} \Phi_X(\mathbf{J}, \omega)$$

We introduce finite-volume Gibbs measures $\mu_\Lambda^{\mathbf{J}, \omega}$ in a finite volume Λ with external conditions given by configuration $\omega \in \Omega$ by

$$\mu_{\mathbf{J}, \Lambda}^{\omega}(f) \equiv \frac{1}{Z_\Lambda(\mathbf{J}, \omega)} \int_{\mathbf{M}^\Lambda} \mu_0^\Lambda(dx_\Lambda) \exp[-\beta U_\Lambda(\mathbf{J}, x_\Lambda \circ \omega_{\Lambda^c})] \, f(x_\Lambda \circ \omega_{\Lambda^c})$$

for local functions f, where μ_0^Λ denotes the product measure on Λ of the probability $\nu_{\mathbf{M}}$ on $\mathbf{M}$ corresponding to the Riemannzan volume to Σ_Λ.

It is proven in ref. 1 that, under the assumption that the interaction has finite exponential bounds [see (2)] and if the temperature is high enough, one can construct an infinite Gibbs measure $\mu_{\mathbf{J}}$ on the whole lattice as a limit of the measures $\mu_{\mathbf{J}, \Lambda_n}^{\omega}$ which is independent of the boundary conditions ω at the countable exhaustion Λ_n of the lattice.

Moreover, we can also define a Markov generator $\mathbf{L}_\Lambda \equiv \mathbf{L}_{\Lambda, \mathbf{J}}$ in a finite volume Λ by setting

$$\mathbf{L}_\Lambda \equiv \sum_{i \in Z^d} \mathbf{L}_i^\Lambda$$

where

$$\mathbf{L}_i^\Lambda \equiv \mathbf{L}_{\mathbf{J}, i}^\Lambda \equiv \Delta_i - \nabla_i H_{i, \Lambda} \cdot \nabla_i$$

with Δ_i denoting the Laplace–Beltrami operator and

$$H_{i, \Lambda} \equiv \sum_{\substack{X \subseteq \Lambda \\ X \ni i}} \Phi_X$$

Then for any $\mathbf{J} \in \mathbb{J}$, we have a well defined Markov semigroup $P_t^\Lambda \equiv P_t^{\mathbf{J}, \Lambda} \equiv e^{t\mathbf{L}_\Lambda}$.

It has been proven in ref. 3 that under the moderate growth assumption (2), one can construct for almost all interaction an infinite volume semigroup $P_t \equiv P_t^{\mathbf{J}}$ defined formally by an operator

$$\mathbf{L} \equiv \mathbf{L}_{\mathbf{J}} \equiv \sum_{i \in \mathbb{Z}^d} \mathbf{L}_i \tag{1}$$

as a limit of the finite-volume semigroups P^{Λ_n}. It was even shown in Theorem 2.1 of ref. 3 that (2) is enough to ensure an approximation property similar to the nonrandom case (i.e., to the case where the couplings are bounded).

3. STATEMENT OF THE RESULTS

Our main result is a fast decay in ferromagnetic random spin systems on a lattice. We show that, under some high-temperature restriction on the interaction, the corresponding Langevin dynamics have a decay to equilibrium much faster than any algebraic with probability one. It is worth noting that this decay is not as fast as the stretched exponential decay proven in ref. 3 for the two-dimensional case and general type of interaction, and that there is no obvious reason why it should be so.

To state the result, we will assume that we are given a ferromagnetic random interaction $\Phi \equiv \{\Phi_X\}_{X \in \mathscr{F}}$ of finite range $R > 0$, i.e., a random interaction such that, for any $i \in \mathbb{Z}^d$,

$$-\nabla_i \sum_{X \ni i} \Phi_X$$

is an increasing function of the spins with respect to the order $\preccurlyeq$ defined by

$$x \preccurlyeq y \leftrightarrow x_i \leqslant y_i \qquad \forall i \in \mathbb{Z}^d$$

Moreover, we will make the following assumption on the random functions $\Phi \equiv \{\Phi_X\}_{X \in \mathscr{F}}$.

(H1) We first assume that the interaction has moderate growth, i.e., that Φ has some exponential moments and more precisely that for a real number ξ large enough, we have

$$\sup_X \mathbb{E} \exp\{\xi \, \|\tilde{\Phi}_X(\mathbf{J}, \cdot)\|_u\} < \infty \tag{2}$$

where $\|\cdot\|_u$ is the uniform norm, and that

$$\sup_X \sup_k \mathbb{E} \exp\{\xi \|\nabla_k \tilde{\Phi}_X(\mathbf{J}, \cdot)\|_u\} < \infty \tag{3}$$

as well as

$$\sup_X \sup_{kj} \mathbb{E} \exp\{\xi \|\nabla_k \nabla_j \tilde{\Phi}_X(\mathbf{J}, \cdot)\|_u\} < \infty \tag{4}$$

(H2) We will then assume that we are in a high-temperature regime, namely that there is a small constant $J_0 > 0$ (specified later) such that

$$0 < p_1 \equiv \sup_X \mathbb{E}\{\|\Phi_X\|_u > J_0\} < p_c^h(d, R) \tag{5}$$

where $p_c^h(d, R)$ is a clustering coefficient (see Kesten[5] and Lemma 5.4 in ref. 3).

We remark that, according to ref. 3, the conditions H1 and H2 show that, with probability one, the exponential approximation property for the infinite volume semigroup holds up to linear time in the size of the finite volume. As a consequence, our Markov process is Feller continuous. The condition H2 ensures that with probability one we have uniqueness and exponential cluster property for equilibrium measures.

Our result is then stated as follows:

Theorem 3.1. Suppose that conditions H1 and H2 are satisfied with J_0 sufficiently small. Then there is a $\theta > 0$ and an $\varepsilon > 0$ such that for any local function $f \in \mathscr{C}^1$, with $\mathbb{E}$ probability one, we have

$$\|P_t^{\mathbf{J}} f - \mu_{\mathbf{J}}(f)\|_u \leqslant C(\Lambda(f), \mathbf{J}) \exp\{-\varepsilon \exp[\theta(\log t)^{1/(d-1)^2}]\} \times \left[\sum_{i \in \mathbb{Z}^d} \mu_{\mathbf{J}}(|\nabla_i f|^2)\right]^{1/2} \tag{6}$$

for all $t \geqslant T(\Lambda(f), \mathbf{J})$, with some positive random variables $C(\Lambda(f), \mathbf{J})$ and $T(\Lambda(f), \mathbf{J})$.

This result is deduced from the following logarithmic Sobolev inequality for finite cubes:

Theorem 3.2. Suppose the conditions of Theorem 3.1 are satisfied. Then for any cube $\Lambda_n \equiv [-n, +n]^d$, for almost all $\mathbf{J} \in \mathbb{J}$, there exists a positive integer $N(\mathbf{J})$ such that, for any $n > N(\mathbf{J})$, for any boundary condition ω,

$$\mu_{\mathbf{J}, \Lambda_n}^{\omega} f^2 \log f^2 \leqslant 2c_{\mathbf{J}, \Lambda_n} \mu_{\mathbf{J}, \Lambda_n}^{\omega}(f(-L_{\Lambda_n} f)) + \mu_{\mathbf{J}, \Lambda_n}^{\omega} f^2 \log \mu_{\mathbf{J}, \Lambda_n}^{\omega} f^2 \tag{7}$$

with

$$c_{J, \Lambda_n} \leqslant c(J) \exp\{\eta(\log n)^{(d-1)^2}\} \tag{8}$$

with some random variable $c(J) \in (0, \infty)$ and η both independent of n, boundary conditions, and the function f.

It should be emphasized that the last theorem holds without any restriction such as ferromagneticity or the choice of the Riemannian manifold M. In the two-dimensional case, Theorem 3.2 has already been proved and even extended to the discrete setting in ref. 3. The authors showed there as well that, if J_0 and p_1 are small enough, a stretched exponential decay to equilibrium occurs. In higher dimension, the bound (8) on the logarithmic Sobolev coefficient grows much faster than any polynomial, so that the strategy followed in ref. 3 is useless. Nevertheless, it appears that Theorem 3.2 is enough to get Theorem 3.1 for ferromagnetic systems.

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REFERENCES

1. H. Von Dreyfus, A. Klein, and J. F. Perez, Taming Griffiths singularities: Infinite differentiability of quenched correlation functions, *Commun. Math. Phys.* **170**:21–39 (1995).
2. G. Gelis and C. Maes, Percolation techniques in disordered spin flip dynamics: Relaxation to the unique invariant measure, K.U. Leuven, Preprint (1995).
3. A. Guionnet and B. Zegarlinski, Decay to equilibrium in random spin system on a lattice, *Commun. Math. Phys.* to appear.
4. A. Guionnet and B. Zegarlinski, Decay to equilibrium in random spin system on a lattice II. Electronic database at Austin: mp_arc@math.utexas.edu.
5. H. Kesten, Aspect of first passage percolation, in *École d'été de St Flour*, (Lecture Notes in Mathematics, No. 1180), (Springer-Verlag, Berlin, 1988), pp. 125–264.
6. F. Martinelli and E. Olivieri, Approach to equilibrium of Glauber dynamics in the one phase region: I. The attractive case/II. The general case, *Commun. Math. Phys.* **161**:447–486, 487–514 (1994).
7. B. Zegarlinski, Ergodicity of Markov semigroups, in *Proceedings of the Conference: Stochastic Partial Differential Equations, Edinburgh 1994*, A. Etheridge, ed. (Cambridge University Press, Cambridge, 1995), pp. 312–337.